Test 3 / Numerical Mathematics 1 / June 5th 2020, University of Groningen

A simple calculator is allowed.
No additional material is allowed.
All answers need to be justified using mathematical arguments.
Total time: 1 hour 30 minutes (time includes upload of the PDF with your answers to Nestor) + 15 minutes (if special needs)
Remember: oral "checks" may be run afterwards.
Grade = (obtained points) + 1

Exercise 1 (4 points)

Consider the ODE for $r(t)$:

$$r'(t) = -c\, r(t)\ , r(0) = r_0 \tag{1}$$

with $c \in \mathbb{R}^+$.

(a) $\boxed{2.0}$ Applying the β-method for the ODE (1) the following algebraic equation is obtained for $r_{k+1} \approx r(t_{k+1})$:

$$r_{k+1} = r_k - hc\beta r_{k+1} - hc(1-\beta)r_k$$

with $h = t_{k+1} - t_k$, $0 \leq \beta \leq 1$. Obtain a condition for h in terms of c, β such that $|r_{k+1}| < |r_k|$. Explain how, in numerical computations, that condition affects the choice of h.

(b) $\boxed{2.0}$ For $c = 2$, $h = 0.1$:
(i) Compute the approximations at $t = h$ using $\beta = 0$, $\beta = 1/2$ and $\beta = 1$.
(ii) Verify that the exact solution of the ODE is $r(t) = r_0 e^{-ct}$.
(iii) Which value of β leads to more accurate results (with respect to the exact solution)?
(iv) Are the results you obtained as expected from the order of accuracy of each of the methods?

Exercise 2 (5 points)

Consider the non-linear ODE for $y(x)$:

$$y' = -y^5 - y^3\ , y(0) = y_0$$

(a) $\boxed{1.0}$ Using one time-step of the forward Euler method ($\beta = 0$), find an expression for $y_1 \approx y(x_1)$, $x_1 > 0$. Approximate y_1 using two Newton iterations. Justify your choice of the initial guess for the Newton iterations.

(b) $\boxed{1.0}$ Using one time-step of the backward Euler method ($\beta = 1$), find an expression for $y_1 \approx y(x_1)$, $x_1 > 0$. Approximate y_1 using one Newton iteration. Use y_0 as initial guess for the Newton iterations.

(c) $\boxed{1.0}$ Show that $\frac{d}{dx}(y^2(x)) \leq 0$.

(d) $\boxed{2.0}$ Show that $(y_1)^2 \leq (y_0)^2$ for $\beta = 1$ and for all $y_0 \in \mathbb{R}$.